# On the "Matrix Approach" to Interacting Particle Systems

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Dedicated to Giovanni Jona-Lasinio on the occasion of his seventieth birthday.

**Abstract**

Derrida et al. and Schütz and Stinchcombe gave algebraic formulas for the correlation functions of the partially asymmetric simple exclusion process. Here we give a fairly general recipe of how to get these formulas and extend them to the whole time evolution (starting from the generator of the process), for a certain class of interacting systems. We then analyze the algebraic relations obtained to show that the matrix approach does not work with some models such as the voter and the contact processes.

*Key words and phrases:* simple exclusion process, matrix product states, open system, stationary non equilibrium states.

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# 1 Introduction

A few years ago Derrida et al.[3],[2] suggested an intriguing "matrix approach" to the one-dimensional Asymmetric Simple Exclusion Process (ASEP). This approach has later been used to treat variants of the model[4],[7],[8], extended to non steady states and by Schütz et al.[12],[13],[16], used to study fluctuations by Derrida et al.[5] and the multispecies case by (among others) Isaev et al. [9].

The main aim of this paper is to study the difficulties that arise in potential applications of the matrix approach to cases in which the nearest neighbor interaction or the particle conservation (both present in the ASEP) are violated. Further light on the applicability of the matrix method is shed by the integrability criterion illustrated by Popkov et al.[16].

In section 2 we provide a general recipe (using the generator of the process) to find the algebra of the matrix formalism associated to both the steady state and the whole dynamics of any one-dimensional interacting system such that at each step the configuration changes only in two adjacent sites. A more complete description, with a pedagogical aim will be given elsewhere[6]. In section 3 we apply the recipe to some important interacting systems such as the contact and voter models and show that the matrix algebra obtained is not useful to treat them.

We will consider only systems in the lattice $\{1, \ldots, N\}$, this is an intrinsic limitation of the matrix approach. The dynamics of an interacting particle system is usually defined by giving the generator of the process, the general form of which can be found for instance in Liggett's book[10].

For example, the generator $\Omega$ of the ASEP, if particles jump one site to the right (left) with rate $p$ ($q = 1 - p$) and enter the lattice from the left

(right) at rate $\alpha$ ($\delta$) and leave it at rate $\gamma$ ($\beta$), is defined by [10]:

$$(\Omega f)(\tau) = \sum_{x=1}^{N-1}[p\tau(x)(1-\tau(x+1))+q\tau(x+1)(1-\tau(x))][f(\tau^{x,x+1})-f(\tau)] + [\alpha(1-\tau(1))+\gamma\tau(1)][f(\tau^{1})-f(\tau)] + [\delta(1-\tau(N))+\beta\tau(N)][f(\tau^{N})-f(\tau)] \quad (1)$$

where $\tau = \{\tau(x)\}_{x=1}^{N}$ is the configuration of the system, $\tau^{x,y}$ is the configuration obtained from $\tau$ by exchanging the content of the sites $x$ and $y$, and $\tau^{x}$ is the configuration obtained from $\tau$ by changing the content of the $x$-th site.

In the following formulas, $|V\rangle\rangle$ is a vector in an (as yet) unspecified linear space equipped with an inner product, $D$ and $E$ linear operators on the same space, $\langle\langle W|$ is an element of the dual space. So $\langle\langle W|A|V\rangle\rangle$ is the inner product generally written as $(\mathbf{W}, A\mathbf{V})$.

The formula of Derrida et al. to write the probability of a given configuration in the stationary state of the ASEP is[1]

$$P_N(\tau_1,...,\tau_N) = \frac{1}{Z_N}\langle\langle W|\prod_{j=1}^{N}[\tau_j D + (1-\tau_j)E]|V\rangle\rangle \ , \quad (2)$$

where $D$, $E$, $|V\rangle\rangle$, $\langle\langle W|$ are matrices and vectors that satisfy

$$\begin{aligned} (\beta D - \delta E)|V\rangle\rangle &= |V\rangle\rangle \ , \\ pDE - qED &= D + E \ , \\ \langle\langle W|(\alpha E - \gamma D) &= \langle\langle W| \end{aligned} \quad (3)$$

and $Z_N$ is a normalization factor.

One can check these formulas provide a sufficient condition for the measure to be stationary by observing they satisfy the recursion relations for the probabilities (first due to Liggett[11]) that relate the probabilities for the system with $K$ sites to the ones for the system with $K-1$ sites[1].

## 2 From the Generator to Matrix Product States

Let us start by re-writing the generator by making use of a formalism borrowed from quantum mechanics. For all $j = 1, ..., N$ let us define the Hilbert space $\mathcal{H}_j := span\left\{|0\rangle_j\,, |1\rangle_j\right\} \cong \mathbb{C}^2$ . Consider the operators $a^+, a^-, n, m$ defined by: $a^+|0\rangle = |1\rangle, a^-|0\rangle = 0\ , n|0\rangle = 0\ , m = \mathbb{I} - n\ , a^+|1\rangle = 0\ , a^-|1\rangle = |0\rangle\ , n|1\rangle = |1\rangle$, where $\mathbb{I}$ is the identity. Interpreting $|0\rangle$ and $|1\rangle$ as empty site and occupied site respectively, the role of $a^+, a^-, n$ as *creation, annihilation, number operators* respectively is rather obvious. The most immediate choice of an explicit expression for the operators and vectors above is $|0\rangle = \binom{0}{1}\ , |1\rangle = \binom{1}{0} a^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, a^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, m = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Now we take the tensor product $\mathrm{H}_N = \bigotimes_{j=1}^N \mathcal{H}_j$ to describe the system on all the $N$ sites.

If we consider for example the ASEP, in this "quantum hamiltonian" formalism[14] the generator is given by

$$\begin{aligned} H = &-\sum_k p(a_k^- a_{k+1}^+ - n_k m_{k+1}) + q(a_k^+ a_{k+1}^- - m_k n_{k+1}) + \\ &\gamma(a_1^- - n_1) + \alpha(a_1^+ - m_1) + \beta(a_N^- - n_N) + \delta(a_N^+ - m_N) \\ &= h_1^\partial + \sum_k h_k + h_N^\partial\ , \end{aligned} \tag{4}$$

where the superscript $\partial$ denotes a boundary term and

$$h_1^\partial = \begin{pmatrix} \gamma & -\alpha \\ -\gamma & \alpha \end{pmatrix},\ h_k = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p & -q & 0 \\ 0 & -p & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ h_N^\partial = \begin{pmatrix} \beta & -\delta \\ -\beta & \delta \end{pmatrix}.$$

For any given operator or vector $b$ in the space $\mathcal{H}_k$ we use the notation $b_k \equiv \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes b \otimes \mathbb{I} \cdots \otimes \mathbb{I}$ with $b$ as $k$-th factor. Using a different $h_\cdot$,

this formulation can be used for any process (like the voter and contact, e.g.) such that the occupation number of each site is either 0 or 1, and such that the dynamics involves a couple of neighboring sites at a time (slight generalizations can be treated as well[7][8]). The generator in (4) is the same as in (1) as can be checked by computing the Dirichlet Form for both and verifying that they coincide (the same holds for processes with different $h$). It is however easier to look closely at each part and see what it does. For instance $a^+a^-$ represents a jump to the right and $nm$ takes into account the complementary event (the particle stays where it is).

We now look for a stationary solution of the master equation

$$|\dot{P(t)}\rangle = H|P(t)\rangle \tag{5}$$

which describes the dynamics of the system by giving the time evolution of the vector of probabilities of configurations, i.e. we look for a distribution $|P_s\rangle$ such that $H|P_s\rangle = 0$.

In order to show where the general idea can be guessed from, let us consider again the case of the ASEP, to show[7] that under special conditions (namely $(\alpha+\beta+\gamma+\delta)(\alpha\beta-\gamma\delta)/(\alpha+\delta)(\beta+\gamma) = p-q$) the stationary state is a product state: $|P_s\rangle = \frac{1}{Z_N}\binom{d}{e}^{\otimes N}$ (where $d = (\alpha+\delta)/(\alpha\beta-\gamma\delta)$ , $e = (\beta+\gamma)/(\alpha\beta-\gamma\delta)$ and the normalization constant is clearly $Z_N = (e+d)^N$). To prove that $H|P_s\rangle = 0$, one should first check

$$h_i\left[\begin{pmatrix} d \\ e \end{pmatrix} \otimes \begin{pmatrix} d \\ e \end{pmatrix}\right] = \begin{pmatrix} d \\ e \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} d \\ e \end{pmatrix} . \tag{6}$$

This makes the sum through which $H$ is defined telescopic (recall that we are omitting the factors of the tensor product on which the operators act trivially as the identity), and since

$$h_1^\partial \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} , \; h_N^\partial \begin{pmatrix} d \\ e \end{pmatrix} = -\begin{pmatrix} -1 \\ 1 \end{pmatrix} ,$$

the cancellation of the first term with the last is assured by the boundary terms.

In other words, $H|P_s\rangle = 0$ would be solved for instance if we had zero for all $i$ in the r.h.s. of (6); but this is too restrictive, so we look for the first non trivial possibility: instead of zero, we impose a "telescopic term". This is inspired by the dynamics, that acts with the same $h_.$ on all couple of adjacent sites, so the generator acts twice on each site. We will now try to make the above approach work for non-product states by imposing a similar telescopic property. The idea is to move into a richer context, substituting the numbers 1, $e$, $d$ appearing in (6) with some time-dependent operators (non commuting and acting on an auxiliary space of generally infinite dimension) $S$, $E$, $D$ to be determined, aiming to get the weights of each possible configuration through a bracket with a couple of vectors $\langle\langle W|$ and $|V\rangle\rangle$ to be introduced in the same space. For instance for a system consisting of a single site we would impose $\langle\langle W|\binom{D}{E}|V\rangle\rangle = \binom{\langle\langle W|D|V\rangle\rangle}{\langle\langle W|E|V\rangle\rangle} = \binom{d}{e}$ and clearly, in the case of a product measure, $\langle\langle W|\binom{D}{E}^{\otimes N}|V\rangle\rangle = \binom{d}{e}^{\otimes N}$. We can also write $H\langle\langle W|\binom{D}{E}^{\otimes N}|V\rangle\rangle = \langle\langle W|H\binom{D}{E}^{\otimes N}|V\rangle\rangle$.

Let us now write $|P\rangle = \frac{1}{Z_N}\langle\langle W|\binom{D}{E}^{\otimes N}|V\rangle\rangle$ for the probability vector and plug it into the master equation (5). Clearly $Z_N = \langle\langle W|C^N|V\rangle\rangle$, with $C = D + E$, that does not depends on time by conservation of probability.

It is easy to show that the master equation (5) is satisfied if the following equalities hold (thanks to the same telescopic cancellation mechanism we used for the product state)

$$\begin{aligned}
&\left(\frac{1}{2}\frac{d}{dt_.} + h_.\right)\begin{pmatrix} D \\ E \end{pmatrix} \otimes \begin{pmatrix} D \\ E \end{pmatrix} = \\
&\quad \begin{pmatrix} D \\ E \end{pmatrix} \otimes \begin{pmatrix} -S \\ S \end{pmatrix} - \begin{pmatrix} -S \\ S \end{pmatrix} \otimes \begin{pmatrix} D \\ E \end{pmatrix} , \\
&\langle\langle W| \left[ (\frac{1}{2}\frac{d}{dt} + h_1^\partial)\begin{pmatrix} D \\ E \end{pmatrix} - \begin{pmatrix} -S \\ S \end{pmatrix} \right] = 0 , \\
&\left[ (\frac{1}{2}\frac{d}{dt} + h_L^\partial)\begin{pmatrix} D \\ E \end{pmatrix} + \begin{pmatrix} -S \\ S \end{pmatrix} \right] |V\rangle\rangle = 0 .
\end{aligned} \tag{7}$$

These are the relations of the matrix algebra of the process. If we chose for example the $h_.$ of the ASEP, these equations take the explicit form of the algebra found by Stinchcombe and Schütz[12],[13] that includes as a special

case the stationary one (3) of Derrida et al. (taking $S = \mathbb{I}$ and putting all the time derivatives equal to zero). With this procedure we can exhibit an algebra for all the models with a dynamics involving only a couple of neighboring sites at a time[14],[15] (see the same works for a classification of the models with different $h_.$). If one found an explicit expression for all the operators and vectors, the model could in principle be solved exactly (provided the algebra is not empty). Unfortunately, this is in general very difficult to accomplish (a purely algebraic treatment can also be used[16]). In the case of the ASEP, thanks to the preservation of the number of particles in the bulk dynamics, the local generator $h_.$ has a block form, with zero entries in the first and last row and column. This special form of $h_.$ is such that in stationary conditions the four equations (7) collapse to just one: (3). But this great simplification may not occur for different models. In many cases the algebra can be empty (or too complicated to deal with), as we are going to show for the contact and voter models. We can say that the method works for the processes, such as the ASEP, the probability measures of which are either product, or a generalization that we can classify as "matrix product measures". If one distinguished only between product and non-product states, the choice would be in general only between a numerical tensor product and a convex combination of as many such products as the cardinality of the configuration space. If the states of a process are matrix product, one can chose to deal again with a single tensor product, thanks to the richer nature of the entries, matrices instead of numbers.

Algebras defined by conditions like (7) are called *Diffusion Algebras* [9].

# 3 The matrix Approach Beyond Simple Exclusion

## 3.1 Exclusion process with double jumps

The method to write the matrix algebra of the process can also be extended to the case of dynamics not limited to neighboring sites, such as for instance the exclusion process with jumps of length two permitted. The generator, in the case of symmetric dynamics, is (up to boundary terms):

$$H = -\sum_k (a_k^- a_{k+1}^+ - n_k m_{k+1}) + (a_k^+ a_{k+1}^- - m_k n_{k+1}) +$$
$$(a_k^- a_{k+2}^+ - n_k m_{k+2}) + (a_k^+ a_{k+2}^- - m_k n_{k+2}) = \sum_k h_k,$$

$$h_. = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For this system we impose the telescopic property to solve the master equation in the following way:

$$\left(\frac{1}{3}\frac{d}{dt}_. + h_.\right)\begin{pmatrix} D \\ E \end{pmatrix} \otimes \begin{pmatrix} D \\ E \end{pmatrix} \otimes \begin{pmatrix} D \\ E \end{pmatrix} = \begin{pmatrix} D \\ E \end{pmatrix} \otimes \begin{pmatrix} -S \\ S \end{pmatrix} \otimes \begin{pmatrix} D \\ E \end{pmatrix} -$$
$$2\begin{pmatrix} -S \\ S \end{pmatrix} \otimes \begin{pmatrix} D \\ E \end{pmatrix} \otimes \begin{pmatrix} D \\ E \end{pmatrix} + \begin{pmatrix} D \\ E \end{pmatrix} \otimes \begin{pmatrix} D \\ E \end{pmatrix} \otimes \begin{pmatrix} -S \\ S \end{pmatrix}$$

which is the same as

$$\frac{1}{3}(2\dot{D}D^2 + D\dot{D}D + D^2\dot{D}) + 0 = -DSD + 2SD^2 - D^2S$$

$$\frac{1}{3}(2\dot{D}DE + D\dot{D}E + D^2\dot{E}) + D^2E - ED^2 = -DSE + 2SDE + D^2S$$

$$\frac{1}{3}(2\dot{D}ED + D\dot{E}D + DE\dot{D}) + DED - ED^2 = DSD + 2SED - DES$$

$$\frac{1}{3}(2\dot{D}E^2 + D\dot{E}E + DE\dot{E}) + 2DE^2 - EDE - E^2D = DSE + 2SE^2 + DES$$

$$\frac{1}{3}(2\dot{E}D^2 + E\dot{D}D + ED\dot{D}) - D^2E - DED + 2ED^2 = -ESD - 2SD^2 - EDS$$

$$\frac{1}{3}(2\dot{E}DE + E\dot{D}E + ED\dot{E}) - DE^2 - EDE = -ESE - 2SDE + EDS$$

$$\frac{1}{3}(2\dot{E}ED + E\dot{E}D + E^2\dot{D}) - DE^2 + E^2D = ESD - 2SED - E^2S$$

$$\frac{1}{3}(2\dot{E}E^2 + E\dot{E}E + E^2\dot{E}) - 0 = ESE - 2SE^2 + E^2S$$

These relations define now a cubic algebra, as opposed to a quadratic one, which is therefore not a Diffusion Algebra in the sense of [9]. Unfortunately algebras of degree higher than two are very difficult to handle (see e.g. Vershik[17]). However algebras of degree higher than two appear e.g. in [16].

## 3.2 Voter and Contact Models

For a description of the voter and contact models see Liggett[10]. It is easy to see that the local generator for the voter model can be written in the form of the r.h.s. of (4) with

$$h_{\cdot} = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

and

$$h_1 = \begin{pmatrix} 0 & -\lambda \\ 0 & \lambda \end{pmatrix} \ , \ h_N = \begin{pmatrix} \mu & 0 \\ -\mu & 0 \end{pmatrix}$$

where $\lambda$ and $\mu$ are the rates for opinion changing in the boundary sites. Notice that there are non zero entries in the first and last row. It is easy to compute that

$$h_{\cdot} \begin{pmatrix} D \\ E \end{pmatrix} \otimes \begin{pmatrix} D \\ E \end{pmatrix} = \begin{pmatrix} -\{D,E\} \\ 2DE \\ 2ED \\ -\{D,E\} \end{pmatrix}$$

and so we can conclude that the algebra and its stationary limit are given by

$$\begin{aligned} \frac{1}{2}(\dot{D}D + D\dot{D}) - \{D,E\} = [S,D] &\longrightarrow \{D,E\} = 0 \\ \frac{1}{2}(\dot{D}E + D\dot{E}) + 2DE = SE + DS &\longrightarrow 2DE = C \\ \frac{1}{2}(\dot{E}D + E\dot{D}) + 2ED = -(SD + ES) &\longrightarrow 2ED = -C \\ \frac{1}{2}(\dot{E}E + E\dot{E}) - \{D,E\} = [E,S] &\longrightarrow \{D,E\} = 0 \end{aligned}$$

Hence in stationary conditions

$$[D,E] = C \equiv D + E, \{D,E\} = 0, \mu D|V\rangle = |V\rangle, \langle W|\lambda E = \langle W|.$$

Notice that the relations are similar to the ones of the ASEP, but there is an additional condition: $D$ and $E$ anticommute.

The local generator of the contact model is

$$h_{\cdot} = \begin{pmatrix} 0 & -\alpha & -\alpha & 0 \\ 0 & \alpha+\beta & 0 & -\alpha \\ 0 & 0 & \alpha+\beta & -\alpha \\ 0 & -\beta & -\beta & 2\alpha \end{pmatrix}$$

so that

$$h\binom{D}{E} \otimes \binom{D}{E} = \begin{pmatrix} -\alpha\{D,E\} \\ (\alpha+\beta)DE - \alpha E^2 \\ (\alpha+\beta)ED - \alpha E^2 \\ -\beta\{D,E\} + 2\alpha E^2 \end{pmatrix}$$

and so we can conclude that the algebra is given by

$$\begin{aligned} \frac{1}{2}(\dot{D}D + D\dot{D}) - \alpha\{D,E\} &= [S,D] \\ \frac{1}{2}(\dot{D}E + D\dot{E}) + (\alpha+\beta)DE - \alpha E^2 &= SE + DS \\ \frac{1}{2}(\dot{E}D + E\dot{D}) + (\alpha+\beta)ED - \alpha E^2 &= -(SD + ES) \\ \frac{1}{2}(\dot{E}E + E\dot{E}) - \beta\{D,E\} + 2\alpha E^2 &= [E,S] \end{aligned}$$

so that in stationary conditions $E^2 = 0,\ [D,E] = C\ , \{D,E\} = 0$ if we assume $\alpha = \beta = 1$.

Clearly these relations define a subalgebra of the one for the voter model.

**Theorem 3.1** In stationary conditions, the algebra of the voter model is empty (and *a fortiori* so is the one of the contact process and so are the ones for the whole time evolution).

**Proof** The algebra is

$$DE = (D+E)/2\ ,\ ED = -(D+E)/2\ ,$$

$$DE = -ED\ ,\ D|V\rangle = \mu|V\rangle\ ,\ \langle W|E = \langle W|\lambda\ .$$

If

$$\vartheta_{\cdot} = D,\ E$$

we get, from the first two conditions

$$\langle W|\prod_{k=1}^{N}\vartheta_k|V\rangle = \langle W|[P(D) + Q(E)]|V\rangle = [P(1/\mu) + Q(1/\lambda)]\langle W|V\rangle$$

with some polynomials $P$ and $Q$; but the third condition (anticommutation) also implies

$$\langle W|\prod_{k=1}^{N}\vartheta_k|V\rangle = (\pm)\langle W|E^m D^n|V\rangle = (\pm)(1/\lambda)^m(1/\mu)^n\langle W|V\rangle$$

where $m+n=N$. The two expressions cannot be equal for all values of $\lambda$ and $\mu$. $\Box$

This shows that, following the recipe of section (2), we cannot use the matrix approach. However, the l.h.s of (7) reflects directly the dynamics of the process and does not depend on the matrix formalism, but the telescopic r.h.s. is only inspired by the nearest neighbor nature of the dynamics and it is more "artificial". In other words, if another way to solve the master equation were developed, some kind of matrix approach could still be productive also for those models that cannot be treated with the current matrix approach illustrated in this paper.


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